Indefinite Metric and Positive Probabilities

K. Just and P. Narramore

Department of Physics, University of Arizona, Tucson AZ 85721

We propose a bound on the mass of every state preparable in particle physics. Thus a realistic cutoff can be permitted although it implies an indefinite metric.

1. Feynman's Discussion

As Feynman explained recently [1], negative "probabilities" may be admitted when they are used with caution. The actually concerned squared norms

$$P_\Psi \equiv (\Psi, \Psi) > 0 , \quad = 0 , \quad > 0 \qquad (1)$$

of all vectors in any (pseudo-)Hilbert space $\mathscr{H}$ must be combined in ways to produce probabilities $P_\Phi > 0$ of *physical* states Φ (and $P_0 = 0$ for the "impossible" $\Phi = 0$ only). Hence it would be best to have a concise name for all the real numbers (1), only some of which truly are probabilities.

Until a better name for those P_Ψ is found, let us in general call them (pseudo-)probabilities, in the case of $P_\Psi \leq 0$ pseudo-probabilities. As a (pseudo-)state let us likewise denote every vector Ψ, whether it denotes a *state* or is unphysical due to $P_\Psi \leq 0$ (or for other reasons). Correspondingly, a definitely unphysical Ψ shall be called a pseudo-state. Then Feynman's explanation can be summarized:

All the (pseudo-)probabilities in (1) of (pseudo-)states Ψ can be useful for calculating the *probabilities* $P_\Phi > 0$ of states Φ. (2)

2. Our Suggestion

The principle (2) has simplified Quantum-Electrodynamics since the work of Gupta and Bleuler; in perturbative gauge theories it is indispensable. In papers to follow, (2) will be applied for two current problems (a realistic cutoff [2] and the "poltergeist" [3] of quantum gravity). As preparation, let us defend the

Thesis: In the *mathematical* space $\mathscr{H}_{\mathrm{phys}}$ of Kugo and Ojima [4], denote any vector by Φ_m when m is the least upper bound of its *mass* spectrum. Define $\bar{M}$ and $\underline{M}$ so that each $\Psi = \Phi_m \neq 0$ with $m < \bar{M}$ makes (1) *positive*, while each Φ_m *preparable* in particle physics has $m \leq \underline{M}$. Then we have $\underline{M} < \bar{M}$.

(3)

Since restrictions on "particle physics" and on "preparable" are necessary, it will be impossible to "deduce" such a thesis. We can merely clarify it and anticipate some questions.

In Sect. 3 we argue that objections to our thesis from cosmology are premature. In Sect. 4 we explain the definition given in (3) for the *mass* of any (pseudo-)state. In Sect. 5 we discuss physical spaces related to (3), and in Sect. 6 the S-matrix in these state spaces. Philosophical problems of minor importance in practice are mentioned in Sect. 7; general questions about our "critical" masses $\bar{M}$ and $\underline{M}$ are answered in Section 8. The assertion $\underline{M} < \bar{M}$ from (3) is justified for incoming states of two bodies in Sect. 9, and for complicated systems in Section 10. There we also review the connection [5] between a cutoff and the characteristic mass of QCD [6].

3. Particles and Cosmology

For formal simplicity, many theories admit pseudo-states Ψ. These may have more energy or more particles than the universe, or outrageous charge densities (proton- instead of neutron-stars). Trying to avoid all Ψ with these or similar features would present complications (and the danger of excluding either too much, or less than is worthwhile). Such Ψ are harmless in classical physics, where one simply ignores them in any application.

In quantum mechanics, the occurrence of pseudo-states Ψ in complete bases raises difficulties; but their omission would cause other contradictions (with experiments and within the theory). Nobody is misled, however, because we know the limitations of that non-relativistic theory. Relativistic particle theory is widely believed to encompass all of physics [7]. The understanding proposed below may be too narrow; and one should not hesitate to attempt extensions. Doing so is unsafe, however, until a theory has reasonably been established in *some* realm.

This especially concerns the use of particle theory in cosmology, from where we expect the strongest objections to (3). When rival theories are thus extrapolated, one may finally ask which yields the best cosmology. Yet one should not reject a new version such as one obeying (3) for the sole reason that it contradicts the extrapolation of a previous version. Throughout this paper, we therefore ignore the Big Bang and the central regions of black holes.

Concerning *quantum gravity*, we thus do not share the prevailing point of view that it is of no interest except for extremely strong curvatures. Conversely, we reject the former contention that only "naively linearized" gravity need be quantized. In other words, we regard as most urgent an understanding of those situations in which nonlinear, but not superstrong, quantum gravity is relevant.

the lowest mass of any pseudo-state $\Psi_m \in \mathcal{H}_{\text{phys}}$ with a pseudo-probability $P_\Psi \leq 0$. Other pseudo-states $\Psi \in \mathcal{H}_{\text{phys}}$ are those which do not obey all super-selection rules. Omitting such unrealistic Ψ, one breaks $\mathcal{H}_{\text{phys}}$ into the *union* of mutually incoherent sectors. This breakup complicates the algebra, but is understood in principle (at least for the electric charge).

While by no means every unpreparable Ψ has a pseudo-probability $P_\Psi \leq 0$, with Feynman [1] we expect that every Ψ with $P_\Psi < 0$ is unpreparable. However, the point of this paper is just that no "ad hoc" postulate is needed for this result. Such a postulate, excluding just the Ψ with $P_\Psi < 0$, would leave us with the set Σ_+ of *all* the $\Psi \in \mathcal{H}_{\text{phys}}$ with positive probabilities (1). Then the S-matrix would not be unitary, because that $\Sigma_+ \subset \mathcal{H}_{\text{phys}}$ is not a subspace. For the same reason, nobody defines $\mathcal{H}_{\text{phys}} \subset \mathcal{H}$ as a subset which is not a linear space.

For the sake of this paper, let us use the term "unpreparable" whenever "unpreparable for any sign of (1)" is meant. By separately examining how this notion affects any pseudo-state of mass m and how (3) does so, we shall below conclude that every Ψ with $P_\Psi < 0$ is already unpreparable for a reason unrelated to P_Ψ. All pseudo-states Ψ with $P_\Psi < 0$ will thus turn out unpreparable; but for clarity we must not presume this result.

4. The Mass of any (Pseudo-)State

As in (3) let us label every $\Psi \in \mathcal{H}_{\text{phys}}$ by the least upper bound of its mass spectrum (not any average). Due to gravity, there is a super-selection rule for the total mass m, saying that m cannot be very uncertain in realistic states. This rule has not been proved as completely as that of the electric charge [8], nor can it yet be formulated as simply. Here we merely need its weak consequence that the mass spectrum of every state Φ_m is bounded (recall that we do not name Φ_m a *state* unless it is preparable). Hence the least upper bound $m < \infty$ of that spectrum may for any state Φ_m be called its mass (many pseudo-states Ψ_m have $m = \infty$).

Disregarding subtleties (about the Ψ_m with $m = \underline{M}$ or $m = \bar{M}$), we may call $\underline{M}$ the highest mass of any state *preparable* in particle physics, and $\bar{M}$

5. Spaces with Bounded Mass Spectra

To simplify further discussions, let us write $\mathcal{H}_M$ for the subspace of all (pseudo-)states $\Psi_m \in \mathcal{H}_{\text{phys}}$ with masses $m \leq M$ (for $m \geq 0$ and $M < \infty$). Here we still define the mass of any pseudo-state as the least upper bound of its spectrum. Of course, for physical reasons we presume that there are complete systems of eigenvectors for the total mass. In (pseudo-)Hilbert spaces, the existence of such bases is not guaranteed by pure mathematics [9].

Each $\mathcal{H}_M$ is a linear space, Poincaré- and PCT-invariant, and specified by a single parameter M. Since the subspaces with bounded mass obey $\mathcal{H}_m \subset \mathcal{H}_M$ for $m < M$, each contains preparable states Φ (at least the vacuum). Even for small $M > 0$, every $\mathcal{H}_M$ also includes many pseudo-states such as *linear* combinations of photon-, neutrino-, electron- and positron-states.

Important for us are the following three properties of all spaces $\mathscr{H}_M$ and of the interval

$$\mathscr{I} \equiv \{M \mid \underline{M} < M < \bar{M}\}: \qquad (4)$$

(a) While every $\mathscr{H}_M$ with $M < \underline{M}$ lacks some preparable states Φ, each with $M > \underline{M}$ contains *all* these Φ (together with many unpreparable Ψ).

(b) While every $\mathscr{H}_M$ with $M > \bar{M}$ has an indefinite metric (some of its Ψ make (1) negative), each with $M < \bar{M}$ shows positive definiteness.

(c) As a consequence of (a) and (b), every $\mathscr{H}_M$ with $M \in \mathscr{I}$ given by (4) contains all preparable states Φ (and tremendously many other $\Psi \in \mathscr{H}_{\mathrm{phys}}$), while its metric is positive definite.

6. Cutoff and Unitarity

Since every space $\mathscr{H}_M \subset \mathscr{H}_{\mathrm{phys}}$ with M in (4) has the property (c), each *state* Φ (preparable in particle physics) has a positive probability (1). Although only individual states are concerned in this conclusion, we hardly could have reached it without the help of the spaces $\mathscr{H}_M$ with bounded mass spectra. Considering just the (pseudo-)states, one might be confused by the fact that all the Ψ_M with $M \in \mathscr{I}$ are unpreparable (whereas all their probabilities (1) are positive). All this requires a nonempty interval (4), hence $\underline{M} < \bar{M}$, and this has been the only assertion made (after the definitions) in (3).

As a state space for particle physics, we may adopt any $\mathscr{H}_M$ with $M \in \mathscr{I}$ given by (4). Taking M slightly above $\underline{M}$ would have the philosophical advantage of leaving $\mathscr{H}_M$ not excessively bigger than required by its symmetries (still having tremendously many unpreparable Ψ). In practice, an M slightly below $\bar{M}$ is preferable (any specific theory provides $\bar{M}$ more rapidly than $\underline{M}$). Deferring the arguments for $\underline{M} < \bar{M}$ to Sect. 9, let us first examine the impact of thesis (3) on the S-matrix.

Concerning this S, we assume its unitarity, Poincaré- and PCT-invariance in the $\mathscr{H}_{\mathrm{phys}}$ of Kugo and Ojima [4]. In $\mathscr{H}_M$ four properties (each widely called the "unitarity") are relevant:

(i) The *unitarity* of S in $\mathscr{H}_M$ follows straight from that in $\mathscr{H}_{\mathrm{phys}}$, since $\mathscr{H}_M$ is a subspace of $\mathscr{H}_{\mathrm{phys}}$ and invariant under S (total mass is conserved). In Sect. 5 we therefore preferred such a space to any

other subset in $\mathscr{H}_{\mathrm{phys}}$ (like Σ_+, the set of the vectors with positive probability).

(ii) The Poincaré- and PCT-*invariance* of S is preserved in $\mathscr{H}_M$ because $\mathscr{H}_{\mathrm{phys}}$ shows this invariance. Hence we insist on this invariance instead of trying to select a smaller set such as the "hidden" $\mathscr{R}$ of Section 7.

(iii) The positive *definiteness* of the metric in $\mathscr{H}_M$ by definition requires $M \le \bar{M}$; thus no higher M should be adopted.

(iv) The physical *completeness* of the states included in $\mathscr{H}_M$ by definition demands $M \ge \underline{M}$; hence a subspace of $\mathscr{H}_M$ would *not* suffice.

7. Philosophical Questions

Evidently, most pseudo-states (unphysical Ψ) *cannot* be identified. Doing so would mean finding that "really realistic" set $\mathscr{R} \subset \mathscr{H}_M$ which contains all states Φ (preparable by Nature or by us), but definitely *no* pseudo-state (unpreparable Ψ). This $\mathscr{R}$ is neither Lorentz-invariant (due to the cosmic background radiation) nor PCT-symmetric (the irreversibility distinguishes incoming and outgoing states absolutely). Superselection rules (known [8] or unknown in detail) further diminish $\mathscr{R}$ in extremely complicated ways.

Since there will be restrictions on $\mathscr{R}$ we cannot yet imagine, this true set of preparable states Φ is hidden from us. Still more amazing is it that physics so far appears unaffected by the extent of such a "philosophical" ignorance. Just this fact greatly simplifies many problems. Even here, the unknown $\mathscr{R}$ will actually remain irrelevant.

Likewise of no practical impact is the distinction between "mixtures" and *pure* (pseudo-)states Ψ. It is generally understood that every mixture containing a finite number of Ψ is unpreparable [10]. Whenever we call a pure pseudo-state Ψ unpreparable, we therefore mean that every mixture is unpreparable to which this Ψ gives any contribution. Without this convention, a too complicated language would be needed.

8. Critical Mass Values

In (3) we have defined $\bar{M}$ as the "border" between the masses $M < \bar{M}$ for which a *positive*

probability (1) of $\Psi = \Phi_M \neq 0$ is assured, and the $M \geq \bar{M}$ for which $\Psi = \Psi_M$ may have $P_\Psi < 0$. Thus we avoid the complicated but purely technical question for the relations of $\bar{M}$ to a cutoff mass Λ and to the poltergeist [3] of quantum gravity (occurring below or above Λ). In any case, the last two masses (important for perturbation theory) will not differ excessively from the $\bar{M}$ decisive here.

Therefore let us simplify the language by tentatively assuming $\bar{M}$ equal to Planck's mass:

$$\bar{M} = M_{\mathrm{Pl}} \equiv (8\pi\, G_{\mathrm{Newton}})^{-1/2} \approx 2.44 \cdot 10^{18}\ \mathrm{GeV} \qquad (5)$$

(the reader may reword our arguments to accommodate nearby values). A positive definiteness in the whole $\mathscr{H}_{\mathrm{phys}}$ would imply $\bar{M} = \infty$ (some authors believe in such a "self-regulation"). Without the preceding projection [4] from the "big" space $\mathscr{H}$ (of any perturbative gauge theory) onto $\mathscr{H}_{\mathrm{phys}}$, however, we would find $\bar{M} = 0$ because the "timelike photons" in $\mathscr{H}$ have $P_\Psi < 0$ and the mass $m = 0$.

Next we must estimate $\underline{M}$, the "highest mass" of the states Φ preparable in particle physics. Since only the *existence* but not the length of the interval (4) is relevant, we solely need to examine processes leading to especially massive states. Whenever a mechanism yields masses clearly below (5), it can be dismissed because it is compatible with our thesis.

9. Binary Collisions

The highest "elementary" mass widely expected belongs to one of the many particles predicted by Grand Unifications. Calling it a "gu" for brevity, let us generously ascribe to it 10^{16} GeV. Leaving aside the question of how even a single gu can be *produced*, we find that a pair of them, colliding with "barely relativistic" speed, gives less than 10^{17} GeV. Thus only extremely relativistic collisions of gus could endanger (3) with (5).

Having found (3) safe even against the last speculations, we need not dwell on events admittedly possible (in the present universe). Intergalactic rays [11], each with less than 10^{13} GeV (otherwise unable to penetrate the cosmic background), in head-on collisions yield masses far below (5). The energies from *accelerators* (to be discovered [12] or built) are not limited by the known laws of physics. These cannot prevent, however, our search for others (quantum gravity and a cutoff). One rather should ask what these *new* laws permit.

Our thesis (3) with (5) indeed effects (the scattering of two incoming) fundamental particles with energies above 10^{18} GeV. We thus know at what *expense* a cutoff may be admitted. There remains the question: "Which theorists will reject a cutoff near (5) in order to keep possible (a) highly relativistic scatterings between gus, (b) colliders with 10^{18} GeV per particle, (c) the current expectations at super-densities (which make gravity a strong interaction), (d) the joint collisions excluded in Section 10?"

For still another cause, one may fail to see that all states preparable in particle physics have bounded mass spectra: There is the widespread idea that it makes sense to ascribe a vector Ψ to "the actual" or even to "every possible" quantum state of the universe. While this idea is evidently unacceptable to us, it would lead us too far to investigate, which theorists favor or reject it and why.

10. Complex Systems

Most collisions at high energies start from *two* particles (three incoming bodies sometimes scatter in chemistry). The existing colliders could actually be more valuable if the 6 quarks of 2 protons would react jointly, but this practically never happens. We still need *not* exclude such "compact" reactions unless they involve outrageously many particles or enormous energies. For instance, we have not seen any proposal for collimating 10^{16} *beams* of protons and anti-protons so that their joint collisions could create a gu. Not even the bosons W and Z are produced in such a way (nor has nuclear fusion been achieved by colliding dust grains).

States with masses above (5) indeed occur in macro-physics. To simplify the language, we do not regard them as objects of *particle* physics. Theorists may dream of describing collisions or at least the ground states of macroscopic (or even celestial) bodies in detail by quantum field theory; yet nobody does so. If one could, we should just rephrase our thesis (3). There persists the *conclusion* that the poltergeist [3] and Pauli-Villars ghosts [2] can be admitted.

An objection opposite to that from cosmology seems to arise in hadron physics, where a mass

$\Lambda_{QCD} \approx 0.1$ GeV is characteristic for QCD [6]. If it is not emphasized that Λ_{QCD} is *not* a cutoff, one might fear that (5) must be greatly lowered. The consistency between (5) and $\Lambda_{QCD} \approx 0.1$ GeV can be understood from the work of Polchinski [5]. Starting from any theory with an arbitrary fixed cutoff mass Λ, he deduces an approximation useful at energies far below Λ. Such an "effective" theory involves a mass μ near those energies for which good accuracy is intended. Hence this μ is compatible with *any* much higher cutoff Λ in the fundamental theory.

[1] R. P. Feynman, Negative Probability, preprint, Pasadena 1985. To appear in "Quantum Theory and Beyond" (essays in Honour of David Bohm), B. G. Hiley and D. Peat (eds.), Routledge & Kegan Paul, London 1987.

[2] K. Just and L. S. The, Top and Higgs Mass from Cancellation Conditions, preprint. Tucson 1986. Z. Naturforsch. **42 a**, 241 (1987).

[3] N. H. Barth and S. M. Christensen, Phys. Rev. **D28**, 1876 (1983).

[4] T. Kugo and I. Ojima, Suppl. Prog. Theor. Phys. **66**, 1 (1979).

[5] J. Polchinski, Nucl. Phys. **B 231**, 269 (1984).

[6] F. J. Ynduráin, Quantum Chromodynamics, Springer, New York 1983.

[7] R. P. Feynman, Theory of Fundamental Processes, Benjamin, New York 1961, p. 1.

[8] F. Strocchi and A. S. Wightman, J. Math. Phys. **15**, 2198 (1974); D. Buchholz, Commun. math. phys. **85**, 49 (1982).

[9] K. L. Nagy, State Vector Spaces with Indefinite Metric in Quantum Field Theory, Noordhoff, Groningen 1966.

[10] R. Blankenbecler and M. Hussein-Partovi, Phys. Rev. Lett. **54**, 373 (1985).

[11] G. Khristiansen, G. Kulikov, and J. Fomin, Cosmic Rays of Superhigh Energies, Thiemig, München 1980.

[12] P. K. Mac Keon and T. C. Weekes, Scient. American, **253**, 60 (November 1985).